Topology and Physics 2019 - lecture 2

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2.1 Maxwell theory in differential form notation

Maxwell's theory of electrodynamics is a great example of the usefulness of differential forms. A nice reference on this topic, though somewhat outdated when it comes to notation, is [1]. For notational simplicity, we will work in units where the speed of light, the vacuum permittivity and the vacuum permeability are all equal to 1: $c = \epsilon_0 = \mu_0 = 1$.

2.1.1 The dual field strength

In three dimensional space, Maxwell's electrodynamics describes the physics of the electric and magnetic fields $\vec{E}$ and $\vec{B}$. These are three-dimensional vector fields, but the beauty of the theory becomes much more obvious if we (a) use a four-dimensional relativistic formulation, and (b) write it in terms of differential forms. For example, let us look at Maxwells two source-free, homogeneous equations:

$$\nabla \cdot B = 0, \qquad \partial_t B + \nabla \times E = 0. \tag{2.1}$$

That these equations have a relativistic flavor becomes clear if we write them out in components and organize the terms somewhat suggestively:

$$\begin{array}{ccccccccccc} 0 & + & \partial_x B^x & + & \partial_y B^y & + & \partial_z B^z & = & 0 \\ -\partial_t B^x & + & 0 & - & \partial_y E^z & + & \partial_z E^y & = & 0 \\ -\partial_t B^y & + & \partial_x E^z & + & 0 & - & \partial_z E^x & = & 0 \\ -\partial_t B^z & - & \partial_x E^y & + & \partial_y E^x & + & 0 & = & 0 \end{array} \tag{2.2}$$

Note that we also multiplied the last three equations by -1 to clarify the structure. All in all, we see that we have four equations (one for each space-time coordinate) which each contain terms in which the four coordinate derivatives act. Therefore, we may be tempted to write our set of equations in more "relativistic" notation as

$$\partial_\mu \hat{F}^{\mu\nu} = 0 \tag{2.3}$$

with $\hat{F}^{\mu\nu}$ the coordinates of an antisymmetric two-tensor (i. e. an antisymmetric *matrix*) that we can write as

$$\hat{F}^{\mu\nu} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -E^z & E^y \\ -B^y & E^z & 0 & -E^x \\ -B^z & -E^y & E^x & 0 \end{pmatrix} \tag{2.4}$$

Some remarks on notations:

- Greek indices $\mu, \nu, \ldots$ always run over (t, x, y, z). These coordinates will also be labeled (x^0, x^1, x^2, x^3), in which case $\mu, \nu, \ldots$ run over $(0, 1, 2, 3)$. To avoid confusing e. g. x^2 for "x-squared", we will usually not write numeric indices explicitly. One exception is in sums, where we will write sums like $\sum_{\mu=0}^{3}$, meaning that all four values of the index are summed over.

- When we need an ordering of the four coordinates, for example to write matrices for components as in (2.4), we will always use the above ordering, with t before x, y and z. Moreover, when writing matrices for components, the first index always labels the row and the second one the column. Thus, for example, $\hat{F}^{ty} = \hat{F}^{02}$ is the component in the first row and the third column in the matrix above, equal to B^y.

- When indices are repeated, once with a lower and once with an upper index, we use the Einstein summation convention: the repeated index is summed over. Thus, $\partial_\mu \hat{F}^{\mu\nu}$ really means $\sum_{\mu=0}^{3} \partial_\mu \hat{F}^{\mu\nu}$.

- When we write the same quantity with upper and lower indices, the two are related by contraction with the components of the metric, $g_{\mu\nu}$ or those of its inverse, $g^{\mu\nu}$. Thus, if we use both X_μ and X^ν, the two are related as $X_\mu = g_{\mu\nu} X^\nu$.

- We will use a "mostly plus" convention for the metric tensor. Thus, in flat space (where we will also use the symbol η for the metric), the metric components are

$$g_{\mu\nu} \equiv \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using the last two remarks, we can also write the content of (2.4) in terms of a lower-index object:

$$\hat{F}_{\mu\nu} = \begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & -E^z & E^y \\ B^y & E^z & 0 & -E^x \\ B^z & -E^y & E^x & 0 \end{pmatrix} \tag{2.5}$$

where accourding to our conventions the entries in the first row and column have changed sign. Why would be interested in this object in particular? Because *lower* index objects can naturally contracted by objects with an upper index like dx^μ to create differential forms. Thus, in this case, we can constructed a *two-form*

$$\hat{F} = \hat{F}_{\mu\nu} dx^\mu \wedge dx^\nu \tag{2.6}$$

Note that in general, since $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$, the above construction only depends on the *antisymmetric* part of the matrix involved. But since our matrix $\hat{F}_{\mu\nu}$ was antisymmetric to begin with, in this case we do not lose any information by turning it into a two-form. This can be viewed as a general rule of thumb: in physics, if you encounter a lower-index object which is antisymmetric, it is probably useful to write it as a differential form!

The 2-form $\hat{F}$ is known in Maxwell theory as the *dual field strength*. The reason for the word "dual" will become clear soon, but for that we first need one more technical ingredient: the Hodge star operator.

2.1.2 The Hodge star

The Hodge star operator is an operator, denoted by $\star$, that maps p-forms on a d-dimensional space to $(d-p)$-forms. Note that for a general d-dimensional space, the number of basis p-forms $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ is $\binom{d}{p} = \binom{d}{d-p}$, so that the number of basis $(d-p)$-forms is the same. Thus, it make sense to try to map one space to the other in a 1-1 fashion, and this is exactly what the $\star$-operator will do for us.

In words, the idea of the Hodge star operator is to take a p-form, strip every component of its $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ part, and replace that part by the remaining $(d-p)$ differentials $dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{d-p}}$. To be more precise, let us first assume that we are in a space with a generic metric $g_{\mu\nu}(x)$; we will soon go back to the simple flat space situation where $g_{\mu\nu} = \eta_{\mu\nu}$. For our construction, we need the antisymmetric ϵ-symbol, defined by

$$\epsilon_{\mu_1 \cdots \mu_d} = \pm 1 \tag{2.7}$$

with a plus sign if $(\mu_1 \cdots \mu_d)$ is an even permutation of $(1 \cdots d-1, 0)$ and a minus sign if it is an odd permutation. Note the slightly odd convention here where the 0 index is placed at the end; we use this convention to adapt to the literature.

On the basis p-forms, the $\star$ operator is now defined as

$$\star \left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \right) = \frac{\sqrt{|g|}}{(d-p)!} \epsilon^{\mu_1 \cdots \mu_p}{}_{\nu_1 \cdots \nu_{d-p}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{d-p}}. \tag{2.8}$$

Here, $|g|$ is the absolute value of the determinant of the metric. On more general forms (i.e. linear combinations of these basis forms with arbitrary x-dependent coefficients), the action of $\star$ is then defined linearly, acting on every basis d-form separately. The prefactors in the above definition are chosen in such a way that the $\star$-operator has several nice properties; we will have to say more about those in a moment.

First, though, let us simplify to the case of our interest: the Hodge star operator in flat space-time. Here, all the coefficients on the right hand side of (2.8) become ± 1, since $|g| = 1$ and the factor $\frac{1}{(d-p)!}$ exactly cancels against the fact that (remember the Einstein convention!) there are $(d-p)!$ equal terms in the sum that follows it. Figuring out the signs is a matter of meticulous bookkeeping; as an exercise, the reader my try to work out for example how the Hodge star works on 2-forms in four flat space-time dimensions:

$$\begin{aligned}
\star(dt \wedge dx) &= -dy \wedge dz & \qquad \star(dy \wedge dz) &= +dt \wedge dx \\
\star(dt \wedge dy) &= +dx \wedge dz & \qquad \star(dx \wedge dz) &= -dt \wedge dy \\
\star(dt \wedge dz) &= -dx \wedge dy & \qquad \star(dx \wedge dy) &= +dt \wedge dz
\end{aligned} \tag{2.9}$$

Let us now mention some of the interesting properties of the Hodge star operator:

- Reading the above table of equations from left to right, we see that in this example, $\star^2 = -1$. In fact, it can be shown that generically, $\star^2 = -(-1)^{p(d-p)}$ for p-forms in a d-dimensional Lorentzian space. (On a Euclidean space, the overall minus sign disappears.) In particular, of course, this means that the Hodge star operator is invertible, with inverse $\pm\star$.

- The Hodge star operator can be used to define an inner product on the space of p-forms. In particular, if α and β are two p-forms, one defines

  $$(\alpha, \beta) = \int_M \alpha \wedge \star\beta. \tag{2.10}$$

  Writing this definition out in components, one finds

  $$(\alpha, \beta) = \frac{1}{p!} \int_M \sqrt{|g|} \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p} \; d^p x. \tag{2.11}$$

  From the latter expression, it is clear that this inner product is symmetric. On a Euclidean manifold, it is moreover positive definite and nondegenerate, but those statements are not true when our manifold has a Lorentzian metric.

- The definition (2.8), in terms of components, may seem somewhat ugly from a mathematical point of view. In fact, the previous bullet point can be used to give a nicer, coordinate independent definition of the Hodge star operator. On a manifold with a metric g, we can define a map from $\Omega^p(M) \times \Omega^p(M)$ to the space of functions on M, $\Omega^0(M)$ as follows:

  $$\langle \alpha, \beta \rangle \equiv \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p}. \tag{2.12}$$

  Moreover, the manifold M has a natural definition of a (top degree) d-form, the *volume form* ω:

  $$\omega \equiv \sqrt{|g|} dx^0 \wedge \cdots \wedge dx^{d-1}. \tag{2.13}$$

When we apply a change of coordinates, some straightforward manipulations show that both of these constructions are in fact independent of our choice of coordinates. As a result, the Hodge star operator satisfies the coordinate independent relation

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \omega \tag{2.14}$$

requiring that for given β this holds for any α actually completely fixes what $\star\beta$ must be. This coordinate independent statement is often used as the definition of the Hodge star operator in the mathematics literature. (Though for many computations, the written out form in (2.8) is often more useful.)

- Using the Hodge star operator, one can define an adjoint operator to the exterior derivative:

$$d^\star \equiv \star d \star, \tag{2.15}$$

that is, we fist turn a p-form into a $(d-p)$-form using the Hodge star operator, then apply the exterior derivative, turning in into a $(d-p+1)$-form, and then "go back" using the Hodge star. We end up with a $p-1$-form, and so the $d^\star$-operator acts in the opposite direction of the d-operator: it lowers the degree of the form instead of raising it. A particularly interesting operator now turns out to be

$$\Delta = dd^\star + d^\star d \tag{2.16}$$

which leaves the degree of the form it operates on unchanged. Writing out all the definitions, one can show that this operator is the familiar Laplacian operator:

$$\Delta = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}. \tag{2.17}$$

The Laplacian operator plays an important role both in physics and mathematics. In fact, it allows us to connect differential geometry to the theory of partial differential equations, leading to a beautiful topic called *Hodge theory*. For now, we will not go into this topic yet, but it may very well appear at some point later in these lectures.

In the above bullet points, we have made many statements without proof. Most of the proofs are not very hard though, and often boil down to index manipulations and precise bookkeeping. Interested readers can try to prove some of these statements themselves, or look up the proofs in e.g. [2].

2.1.3 The field strength

Now that we have introduced the Hodge star operator, let us go back to our subject of study: Maxwell's theory of electromagnetism. In (2.6) we defined a two-form that we called the dual field strength. That terminology already makes it clear that it may be good to view $\hat{F}$ as the Hodge dual of another two-form:

$$\hat{F} = \star F, \qquad \text{or} \qquad F = - \star \hat{F} \tag{2.18}$$

where in the second statement we used the fact that the Hodge star operator squares to -1 when acting on two-forms in four dimensions. In flat space, it is not to hard to write out the components $F_{\mu\nu}$ of this new two-form in a matrix: using (2.9) we easily find

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E_x & 0 & -B^z & B^y \\ -E_y & B^z & 0 & -B^x \\ -E_z & -B^y & B^x & 0 \end{pmatrix} \tag{2.19}$$

Now, we finally have reached the point where we can enjoy the fruits of our labor. First of all, let us compute the three form dF in terms of components. Writing out, for example, the component of this three-form multiplying $dx \wedge dy \wedge dz$, we find

$$(dF)_{xyz} = \partial_x F_{yz} + \partial_y F_{zx} + \partial_z F_{xy} = -\partial_x B^x - \partial_y B^y - \partial_z B^z. \tag{2.20}$$

Up to a sign, we find back the left hand side of the first equation in (2.2). Similarly, writing out the component multiplying $dt \wedge dx \wedge dy$, we find

$$(dF)_{txy} = \partial_t F_{xy} + \partial_x F_{yt} + \partial_y F_{tx} = -\partial_t B^z - \partial_x E^y + \partial_y E^x. \tag{2.21}$$

This gives us the last line in (2.2). The reader can check that the other two components of dF likewise represent the left hand sides of the two remaining equations in (2.2). Thus, we have now found a very simple way to rewrite the homogeneous Maxwell equations: simply as

$$dF = 0. \tag{2.22}$$

This may not seem like a big improvement over our original way of writing the homogeneous Maxwell equations as $\partial_\mu \hat{F}^{\mu\nu} = 0$ in (2.3), but there are two crucial advantages of this new way of writing our equation. First of all, note that the form (2.22) is completely independent of a choice of coordinates on our spacetime manifold M; it simly says that the exterior derivative of a 2-form (both concepts which can be defined without ever referring to coordinates) vanishes. This is in sharp contrast to (2.3), where for example derivatives with respect to the coordinates x^μ appear explicitly.

A second advantage follows from this: once we have written our equations in differential form notation, it will be much easier to see which concepts in Maxwell theory depend on the *geometry* of space-time (even though at this point, that geometry is still mostly flat Minkowski space) and which concepts are actually *topological*.

Before going into examples of this, let us recall that there are two more equations of motion in Maxwell theory: the inhomogeneous equations

$$\nabla \cdot E = \rho, \qquad \nabla \times B - \frac{\partial E}{\partial t} = j. \tag{2.23}$$

In contrast to the two equations in (2.1), these equations have "background source terms" on the right hand side: a charge density scalar ρ and a current density vector j. It is probably not surprising that these two quantities fit together nicely into a four-vector with components J^μ. In exercise 2 below, we will see that these equations also have a nice representation in differential form notation: they can be written as

$$d \star F = \star J \tag{2.24}$$

Note that in this equation, the Hodge star operator appears, which *does* depend on the metric. In other words, even though the expression above depends on specific coordinates, the relation between F and $\star F$ does depend on a choice of metric on space-time.

2.2 Maxwell theory and topology

2.2.1 The electromagnetic potential

To begin exploring the advantages of the differential form notation, let us once again consider the homogeneous Maxwell equation

$$dF = 0. \tag{2.25}$$

This equation simply states that F is a *closed* 2-form. However, if we are in flat, topologically trivial Minkowski space, Poincaré's lemma tells us that this must imply that

$$F = dA \tag{2.26}$$

for some 1-form $A = A_\mu dx^\mu$. Writing this out in components, we find that these equations imply that

$$\vec{E} = \partial_t \vec{A} - \vec{\nabla} A_0, \qquad \vec{B} = -\vec{\nabla} \times \vec{A}. \tag{2.27}$$

In other words, this is the familiar way to write the electric and magnetic fields in terms of a magnetic vector potential $\vec{A}$ and an electric scalar potential $\Phi \equiv A_0$. The derivation of the existence of those potentials has now become extremely simple – it is simply an application of Poincaré's lemma.

2.2.2 Gauge symmetry

Let us for the moment stay in the situation where space-time is topologically trivial. Then, the fact that $F = dA$ has an almost trivial consequence: since $d^2 = 0$, F does not change if we change A to $A + d\Lambda$, for some 0-form Λ. Written out in components, and recalling that we wrote $A_0 = \Phi$, this means that we can change

$$\begin{aligned} \vec{A} &\rightarrow \vec{A} + \vec{\nabla}\Lambda \\ \Phi &\rightarrow \Phi + \partial_t \Lambda. \end{aligned} \tag{2.28}$$

This result is well-known; it is the *gauge symmetry* of the vector potential in electromagnetism. Indeed, it is not too hard to show that $\vec{E}$ and $\vec{B}$ in (2.27) do not change under (2.28). Once again, the derivation of this classical result is a complete triviality once we write electromagnetism in differential form notation.

2.2.3 Intermezzo: Euler-Lagrange equations

In these lectures, we shall use the path integral formalism to compute several physical quantities of interest. A path integral is a way to write expectation values of operators in quantum field theory in terms of a functional integral. The rigorous mathematical definition of path integrals is an intricate subject, but it turns out that using intuition about ordinary integrals, one can work with path integrals pretty much without ever needing full mathematical rigor. Since for us, the path integral is simply a useful tool, not a goal in itself, this is the approach we shall take here.

In (quantum) mechanics, one studies quantities like position and momentum as functions of time: $x(t)$, $p(t)$. A much more general setting is that of (quantum) *field theory*, where the quantities of interest not only depend on the time t, but also on the three position coordinates x^i – or more relativistically written: on four space-time coordinates x^μ. Examples would be quantities like the electric field $E(x,t)$ and the magnetic field $B(x,t)$. Let us denote a generic field by $\phi(x^\mu)$. In quantum field theory, the main bookkeeping device is now the *partition function*, usually denoted by Z:

$$Z = \int D\phi(x,t) \exp\left(\frac{i}{\hbar}\int L[\phi, \partial_\mu \phi] d^4 x\right) \tag{2.29}$$

Here, L (the *Lagrangian*) is a function of the field ϕ and its derivatives $\partial_\mu \phi$, which in most simple situations can be calculated by subtracting the expression for the potential energy from the expression for the kinetic energy. (Note the fact that we are subtracting here: we are not doing the intuitively natural thing which would be to *add* the two energies!) Now, the path integral is an integral over all field configurations with fixed boundary conditions – for example, we may again fix the configuration of the field at given times[1] t_i and t_f.

Depending on taste, the above expression is either mathematics at its worst or physics at its best. The reason is that the space we want to integrate over – some space of 'all field configurations with given boundary conditions' is uncountably infinite dimensional, and therefore properly defining the integral over this space is impossible in almost every situation. We will get back to the mathematical issues with properly defining a path integral in a later lecture.

From a physics perspective, the nice thing is that even tough the path integral is very ill-defined, we can still do very meaningful computations with it! The reason for this is that there are many properties of ordinary integrals – partial integration, for example – that we may expect to also be properties of the path integrals. Rather than attempting to actually *evaluate* path integrals, we will usually work with these properties alone to derive interesting results.

[1] Usually, one chooses the spatial coordinates x^i to run from $-\infty$ to $+\infty$, but strictly speaking one also needs to impose boundary conditions there. This is often left unmentioned, but usually one implicitly chooses boundary conditions where e.g. the fields ϕ fall off fast enough as one goes to spatial infinity.

As an example of this philosophy, let us see how we can obtain classical equations of motion from a path integral. Note that the path integral expression for the partition function,

$$Z = \int D\phi(x,t) \exp\left(\frac{i}{\hbar}\int L[\phi, \partial_\mu \phi] d^4x\right), \tag{2.30}$$

is an integral of the *exponential* of a functional[2]. In the case of ordinary integration, it is well known that such integrals can be computed using a steepest descent procedure. In particular, the main contribution from such an integral comes from the *stationary points* of the exponent of the integrand. (Loosely speaking, the reason for this is that the integrand either oscillates rapidly or decays rapidly away from such stationary points.) Therefore, let us investigate what the stationary *trajectories* are for our path integral - these are the particular paths that we expect to give the main contributions to its value.

A stationary point of a *function* is a point where all of its derivatives vanish. So similarly, let us investigate for which $\phi(x,t)$ the functional

$$S = \int L[\phi, \partial_\mu \phi] d^4x \tag{2.31}$$

does not change to first order under a change in ϕ. That is, we change

$$\phi(x,t) \to \phi(x,t) + \delta\phi(x,t) \tag{2.32}$$

Under such a change, S changes as

$$\delta S = \int \left(\frac{\partial L}{\partial \phi(x,t)}\delta\phi(x,t) + \frac{\partial L}{\partial \partial_\mu \phi(x,t)}\partial_\mu \delta\phi(x,t)\right) d^4x. \tag{2.33}$$

Now, $\delta\phi = 0$ at the boundaries of integration (remember we fixed the boundary conditions once and for all), so we can do a partial integration in the second term of the above expression without picking up any boundary terms. This gives

$$\delta S = \int \left(\frac{\partial L}{\partial \phi(x,t)} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi(x,t)}\right) \delta\phi(x,t) d^4x. \tag{2.34}$$

Now, recall that we want to figure out when S is invariant under *any* change in the field ϕ: δS must be zero for any $\delta\phi(x,t)$. This is only possible if we are at a field configuration $\phi(x,t)$ such that

$$\frac{\partial L}{\partial \phi(x,t)} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi(x,t)} = 0. \tag{2.35}$$

This equation is known as the *Euler-Lagrange equation*; it determines which field configurations contribute most to the path integral. These field configurations have a very clear

[2] A "functional" is simply a function of a function: an object which assigns a number to a given function. In the path integral, $\int L[\phi, \partial_\mu \phi] d^4x$ (usually denoted by S and called the *action*) is such a functional.

physical meaning: they are the solutions to the *classical* equations of motion of the system under consideration. That this must be the case can be seen from the fact that S in the exponential is multiplied by $1/\hbar$: if we make Planck's constant $\hbar$ very small, the oscillations in the integrand will become bigger and bigger, and therefore the contributions of the stationary points will become more and more important. As a result, in the "classical" limit, only these particular field configurations play a role.

2.2.4 The Maxwell action and its equatons of motion

To do quantum physics, we want to view Maxwell's equations as the Euler-Lagrange equations of a specific Lagrangian. This Lagrangian can be derived using the canonical procedure of subtracting the potential energy in the electric and magnetic fields from the kinetic energy. We will not go through this procedure here, but simply write down the answer: the action for Maxwell theory is

$$S = \frac{1}{g}\int_M F \wedge \star F + A \wedge \star J \tag{2.36}$$

where g is a normalization constant ("coupling constant") that is irrelevant if we want to calculate the equations of motion, but that plays a role in the full quantum theory, where we are also interested in the contributions to the path integral of paths that are *not* solutions to the equation of motion. In one of the exercises, we will show that the two inhomogeneous Maxwell equations are indeed the Euler-Lagrange equations for this action. Of course, we only need to be able to derive the inhomogeneous equations, as the homogeneous equations follow automatically from the fact that $dF = 0$.

Given the building blocks that we have (a one-form A and a two-form F), and that we need a four-form to integrate over the entire space-time and obtain a coordinate-independent action, the above formula in fact gives one of the most general "natural" actions that one can write down. For example, a factor of $A \wedge A$ would lead to a vanishing term, as the wedge product is antisymmetric on one-forms. On two-forms, the wedge product is actually symmetric, so at first sight one might wonder if a term of the form

$$\theta \int_M F \wedge F \tag{2.37}$$

would give an interesting contribution to the action. As it turns out, this is not the case for Maxwell theory, where $dF = 0$, as one can write this term as

$$\theta \int_M d(F \wedge A). \tag{2.38}$$

Then, using Stokes' theorem, and using the fact that our fields fall off at infinity, we see that this term actually vanishes. Later on, we will see that in other theories, a term of the above form *does* play an important role though.

2.2.5 Measuring electric and magnetic charge

So far, we have studied situations where space-time is the topologically trivial Minkowski space. Of course, the topological methods we study in this course are most interesting in topologically nontrivial situations. We could study sitiuations where space (or space-time) is taken to be e.g. a sphere or torus, but it is not immediately clear how physical those situations are.

A perhaps more physical topologically non-trivial space occurs when we remove a single point from space. We can think of this point as the position of a particle, where the quantum fields may be discontinuous or even singular. One example of a situation where such a configuration is very interesting is that of a magnetic monopole (which, by the way, has *not* been observed in nature).

Let us begin by looking at an electrically charged particle of charge q_e. Note from equation (2.5) that in the two-form $\star F$, E_x multiplies $dy \wedge dz$, and similary if we cyclically permute (x, y, z). In other words, if we integrate $\star F$ over some *spatial* surface, we are measuring the electric flux through that surface. Through a *closed* surface, the total electric flux measures the charge inside that surface. That is, we have

$$\int_{S^2} \star F = q_e \tag{2.39}$$

Now, let us ask the following question: can we also describe a *magnetically* charged particle in this way? Since the Hodge star operator exchanges E- and B-fields, such a particle should have a flux

$$\int_{S^2} F = q_m. \tag{2.40}$$

Since in Minkowski space S^2 is the boundary of a 3-ball B^3, we get a contradiction, as applying Stokes' theorem to the left hand side of this equation, we get

$$\int_{B^3} dF = q_m, \tag{2.41}$$

but this is cleary a contradiction with $dF = 0$.

Therefore, to be able to construct a magnetically charged particle in our theory, we have to make sure that our S^2 is *not* the boundary of another submanifold. The easiest way to do this is to remove a single point from the interior of S^2. We study this situation in exercise 3, where we shall see that indeed it is now possible to create a magnetically charged particle: the Dirac monopole.

References

[1] G. A. Deschamps, "Electromagnetics and differential forms," in Proceedings of the IEEE, vol. 69, no. 6, pp. 676-696, June 1981.

[2] M. Nakahara, "Geometry, Topology and Physics," Institute of Physics Publishing, 2003.